Algebraic geometry 2

Exercise Sheet 1

PD Dr. Maksim Zhykhovich Summer Semester 2026, 16.04.2026

Exercise 1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X. Let $x \in X$. Show that the induced map on stalks

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x,\ (U, s) \mapsto (U, \varphi_U(s)),$$

is well-defined and is a homomorphism of abelian groups.

Exercise 2. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X.
(1) Assume that $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open $U \subset X$. Show that φ is injective (see Definition 1.12).
Give an example of X and $\varphi : \mathcal{F} \to \mathcal{G}$ for which the inverse statement is false.
(2) Assume in addition that $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Show that in this case φ is injective if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open $U \subset X$.
(3) Assume that $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for every open $U \subset X$. Show that φ is surjective (see Definition 1.12).
Give an example of X and $\varphi : \mathcal{F} \to \mathcal{G}$ for which the inverse statement is false.

Exercise 3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X.
(1) Check that $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, $\operatorname{Coker}\varphi$ from Definition 1.13 are well-defined presheaves with the restriction maps induces from $\mathcal{F}$ or $\mathcal{G}$.
(2) Assume that $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Show that the presheaf $\operatorname{Ker}\varphi$ is a sheaf.